8-3 Graphing Rational Functions

Main Ideas

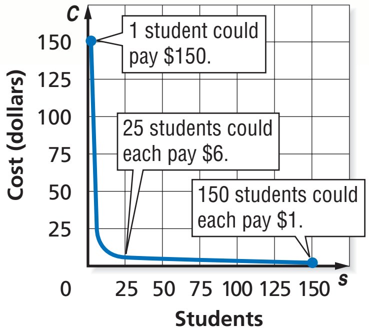
- Determine the limitations on the domains and ranges of the graphs of rational functions.
- Graph rational functions.

New Vocabulary

rational function
continuity
asymptote
point discontinuity

GET READY for the Lesson

A group of students want to get their favorite teacher, Mr. Salgado, a retirement gift. They plan to get him a gift certificate for a weekend package at a lodge in a state park. The certificate costs \$150. If c represents the cost for each student and s represents the number of students, then $c = \frac{150}{s}$.



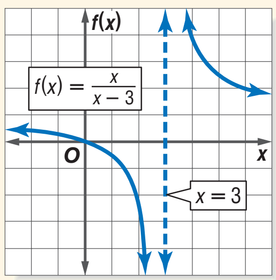
Domain and Range The function $c = \frac{150}{s}$ is a rational function. A **rational function** has an equation of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$. Here are other rational functions.

$$f(x) = \frac{x}{x+3} \qquad g(x) = \frac{5}{x-6} \qquad h(x) = \frac{x+4}{(x-1)(x+4)}$$

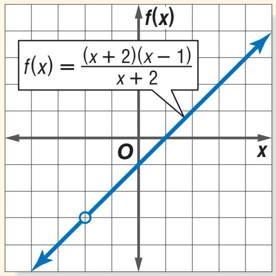
No denominator in a rational function can be zero because division by zero is not defined. The functions above are not defined at $x = -3$, $x = 6$, and $x = 1$ and $x = -4$, respectively. The domain of a rational function is limited to values for which the function is defined.

The graphs of rational functions may have breaks in **continuity**. This means that, unlike polynomial functions, which can be traced with a pencil never leaving the paper, not all rational functions are traceable. Breaks in continuity occur at values that are excluded from the domain. They can appear as vertical asymptotes or as point discontinuity. An **asymptote** is a line that the graph of the function approaches, but never touches. **Point discontinuity** is like a hole in a graph.

KEY CONCEPT — *Vertical Asymptotes*

Property	Words	Example	Model
Vertical Asymptote	If the rational expression of a function is written in simplest form and the function is undefined for $x = a$, then the line $x = a$ is a vertical asymptote.	For $f(x) = \frac{x}{x-3}$, the line $x = 3$ is a vertical asymptote.	

KEY CONCEPT — *Point Discontinuity*

Property	Words	Example	Model
Point Discontinuity	If the original function is undefined for $x = a$ but the rational expression of the function in simplest form is defined for $x = a$, then there is a hole in the graph at $x = a$.	$f(x) = \frac{(x+2)(x-1)}{x+2}$ can be simplified to $f(x) = x - 1$. So, $x = -2$ represents a hole in the graph.	

Concepts in Motion
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EXAMPLE Limitations on Domain

1 **Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of $f(x) = \frac{x^2 - 1}{x^2 - 6x + 5}$.**

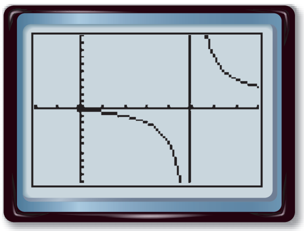
First factor the numerator and denominator of the rational expression.

$$\frac{x^2 - 1}{x^2 - 6x + 5} = \frac{(x-1)(x+1)}{(x-1)(x-5)}$$

The function is undefined for $x = 1$ and $x = 5$. Since $\frac{\cancel{(x-1)}^{1}(x+1)}{\cancel{(x-1)}_{1}(x-5)} = \frac{x+1}{x-5}$,

$x = 5$ is a vertical asymptote, and $x = 1$ represents a hole in the graph.

CHECK You can use a graphing calculator to check this solution. The graphing calculator screen at the right shows the graph of $f(x)$. The graph shows the vertical asymptote at $x = 5$. It is not clear from the graph that the function is not defined at $x = 1$. However, if you use the value function of the CALC menu and enter 1 at the X= prompt, you will see that no value is returned for Y=. This shows that $f(x)$ is not defined at $x = 1$.



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Study Tip

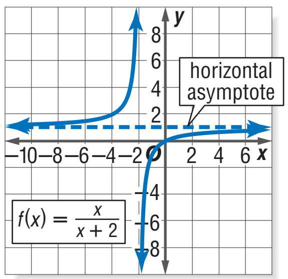
Parent Function

The parent function for the family of rational, or reciprocal, functions is $y = \frac{1}{x}$.

CHECK Your Progress

1. Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of $f(x) = \frac{x^2 + 6x + 8}{x^2 - 16}$.

For some rational functions, the values of the range are limited. Often a horizontal asymptote occurs where a value is excluded from the range. For example, 1 is excluded from the range of $f(x) = \frac{x}{x+2}$. The graph of $f(x)$ gets increasingly close to a horizontal asymptote as x increases or decreases.



Graph Rational Functions You can use what you know about vertical asymptotes and point discontinuity to graph rational functions.

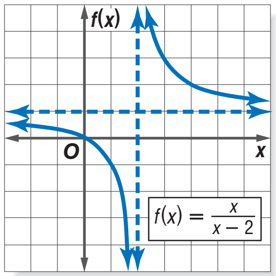
EXAMPLE 2 Graph with Vertical and Horizontal Asymptotes

Graph $f(x) = \frac{x}{x-2}$.

The function is undefined for $x = 2$. Since $\frac{x}{x-2}$ is in simplest form, $x = 2$ is a vertical asymptote. Draw the vertical asymptote. Make a table of values. Plot the points and draw the graph.

As $|x|$ increases, it appears that the y-values of the function get closer and closer to 1. The line with the equation $f(x) = 1$ is a horizontal asymptote of the function.

x	$f(x)$
−50	0.96154
−20	0.90909
−10	0.83333
−2	0.5
−1	0.33333
0	0
1	−1
3	3
4	2
5	1.6667
10	1.25
20	1.1111
50	1.0417



Study Tip

Graphing Rational Functions

Finding the x- and y-intercepts is often useful when graphing rational functions.

CHECK Your Progress

2. Graph $f(x) = \frac{x+1}{x-1}$.

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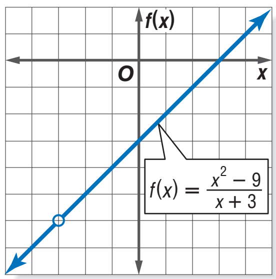
As you have learned, graphs of rational functions may have point discontinuity rather than vertical asymptotes. The graphs of these functions appear to have holes. These holes are usually shown as circles on graphs.

EXAMPLE 3 Graph with Point Discontinuity

Graph $f(x) = \frac{x^2-9}{x+3}$.

Notice that $\frac{x^2-9}{x+3} = \frac{(x+3)(x-3)}{x+3}$ or $x - 3$.

Therefore, the graph of $f(x) = \frac{x^2-9}{x+3}$ is the graph of $f(x) = x - 3$ with a hole at $x = -3$.



CHECK Your Progress

3. Graph $f(x) = \frac{x^2+4x-5}{x+5}$.

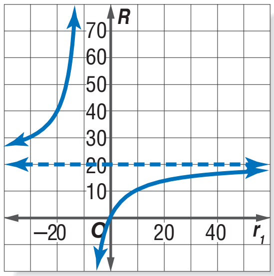
In the real world, sometimes values on the graph of a rational function are not meaningful.

Real-World EXAMPLE — Use Graphs of Rational Functions

4 **AVERAGE SPEED** **A boat traveled upstream at r_1 miles per hour. During the return trip to its original starting point, the boat traveled at r_2 miles per hour. The average speed for the entire trip R is given by the formula $R = \frac{2r_1r_2}{r_1 + r_2}$.**

a. Let r_1 be the independent variable and let R be the dependent variable. Draw the graph if $r_2 = 10$ miles per hour.

The function is $R = \frac{2r_1(10)}{r_1 + 10}$ or $R = \frac{20r_1}{r_1 + 10}$. The vertical asymptote is $r_1 = -10$. Graph the vertical asymptote and the function. Notice that the horizontal asymptote is $R = 20$.



b. What is the R-intercept of the graph? The R-intercept is 0.

c. What domain and range values are meaningful in the context of the problem?

In the problem context, the speeds are nonnegative values. Therefore, only values of r_1 greater than or equal to 0 and values of R between 0 and 20 are meaningful.

CHECK Your Progress

4. **SALARIES** A company uses the formula $S(x) = \frac{45x + 25}{x + 1}$ to determine the salary in thousands of dollars of an employee during his xth year. Draw the graph of $S(x)$. What domain and range values are meaningful in the context of the problem? What is the meaning of the horizontal asymptote for the graph?

CHECK Your Understanding

Example 1 (p. 458)

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of each rational function.

1. $f(x) = \frac{3}{x^2 - 4x + 4}$

2. $f(x) = \frac{x - 1}{x^2 + 4x - 5}$

Graph each rational function.

Example 2 (p. 459)

3. $f(x) = \frac{x}{x + 1}$

4. $f(x) = \frac{6}{(x - 2)(x + 3)}$

5. $f(x) = \frac{4}{(x - 1)^2}$

6. $f(x) = \frac{x - 5}{x + 1}$

Example 3 (p. 459)

7. $f(x) = \frac{x^2 - 25}{x - 5}$

8. $f(x) = \frac{x + 2}{x^2 - x - 6}$

Example 4 (p. 460)

ELECTRICITY **For Exercises 9–12, use the following information.**
The current I in amperes in an electrical circuit with three resistors in series is given by the equation $I = \frac{V}{R_1 + R_2 + R_3}$, where V is the voltage in volts in the circuit and R_1, R_2, and R_3 are the resistances in ohms of the three resistors.

9. Let R_1 be the independent variable, and let I be the dependent variable. Graph the function if $V = 120$ volts, $R_2 = 25$ ohms, and $R_3 = 75$ ohms.

10. Give the equation of the vertical asymptote and the R_1- and I-intercepts of the graph.

11. Find the value of I when the value of R_1 is 140 ohms.

12. What domain and range values are meaningful in the context of the problem?

Exercises

HOMEWORK HELP

For Exercises	See Examples
13–16	1
17–26	2
27, 28	3
29–36	4

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of each rational function.

13. $f(x) = \frac{2}{x^2 - 5x + 6}$

14. $f(x) = \frac{4}{x^2 + 2x - 8}$

15. $f(x) = \frac{x + 3}{x^2 + 7x + 12}$

16. $f(x) = \frac{x - 5}{x^2 - 4x - 5}$

Graph each rational function.

17. $f(x) = \frac{1}{x}$

18. $f(x) = \frac{3}{x}$

19. $f(x) = \frac{1}{x + 2}$

20. $f(x) = \frac{-5}{x + 1}$

21. $f(x) = \frac{x}{x - 3}$

22. $f(x) = \frac{5x}{x + 1}$

23. $f(x) = \frac{-3}{(x - 2)^2}$

24. $f(x) = \frac{1}{(x + 3)^2}$

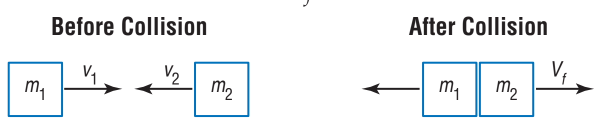
25. $f(x) = \frac{x + 4}{x - 1}$

26. $f(x) = \frac{x - 1}{x - 3}$

27. $f(x) = \frac{x^2 - 36}{x + 6}$

28. $f(x) = \frac{x^2 - 1}{x - 1}$

PHYSICS **For Exercises 29–32, use the following information.**
Under certain conditions, when two objects collide, the objects are repelled from each other with velocity given by the equation $V_f = \frac{2m_1v_1 + v_2(m_2 - m_1)}{m_1 + m_2}$. In this equation m_1 and m_2 are the masses of the two objects, v_1 and v_2 are the initial speeds of the two objects, and V_f is the final speed of the second object.



29. Let m_2 be the independent variable, and let V_f be the dependent variable. Graph the function if $m_1 = 5$ kilograms and $v_1 = 15$ meters per second, and $v_2 = 20$ meters per second.

30. Use the equation and the values in Exercise 29 to determine the final speed if $m_2 = 20$ kilograms.

31. Give the equation of any asymptotes and the m_2- and V_f-intercepts of the graph.

32. What domain and range values are meaningful in the context of the problem?

BASKETBALL **For Exercises 33–36, use the following information.**
Zonta plays basketball for Centerville High School. So far this season, she has made 6 out of 10 free throws. She is determined to improve her free-throw percentage. If she can make x consecutive free throws, her free-throw percentage can be determined using $P(x) = \frac{6 + x}{10 + x}$.

33. Graph the function.

34. What part of the graph is meaningful in the context of the problem?

35. Describe the meaning of the y-intercept.

36. What is the equation of the horizontal asymptote? Explain its meaning with respect to Zonta's shooting percentage.

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of each rational function.

37. $f(x) = \frac{x^2 - 8x + 16}{x - 4}$

38. $f(x) = \frac{x^2 - 3x + 2}{x - 1}$

Graph each rational function.

39. $f(x) = \frac{3}{(x - 1)(x + 5)}$

40. $f(x) = \frac{-1}{(x + 2)(x - 3)}$

41. $f(x) = \frac{x}{x^2 - 1}$

42. $f(x) = \frac{x - 1}{x^2 - 4}$

43. $f(x) = \frac{6}{(x - 6)^2}$

44. $f(x) = \frac{1}{(x + 2)^2}$

45. $f(x) = \frac{x^2 + 6x + 5}{x + 1}$

46. $f(x) = \frac{x^2 - 4x}{x - 4}$

HISTORY **For Exercises 47–49, use the following information.**
In Maria Gaetana Agnesi's book *Analytical Institutions*, Agnesi discussed the characteristics of the equation $x^2y = a^2(a - y)$, the graph of which is called the "curve of Agnesi." This equation can be expressed as $y = \frac{a^3}{x^2 + a^2}$.

47. Graph $f(x) = \frac{a^3}{x^2 + a^2}$ if $a = 4$.

48. Describe the graph. What are the limitations on the domain and range?

49. Make a conjecture about the shape of the graph of $f(x) = \frac{a^3}{x^2 + a^2}$ if $a = -4$. Explain your reasoning.

EXTRA PRACTICE
See pages 908, 933.
Math Online
Self-Check Quiz at
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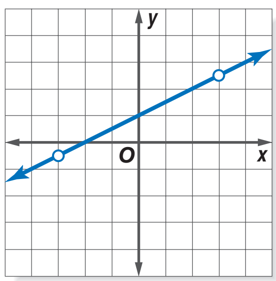
H.O.T. Problems

50. OPEN ENDED Write a function the graph of which has vertical asymptotes located at $x = -5$ and $x = 2$.

51. REASONING Compare and contrast the graphs of $f(x) = \frac{(x - 1)(x + 5)}{x - 1}$ and $g(x) = x + 5$.

52. CHALLENGE Write a rational function for the graph at the right.



53. CHALLENGE Write three rational functions that have a vertical asymptote at $x = 3$ and a hole at $x = -2$.

54. *Writing in Math* Use the information on page 457 to explain how rational functions can be used when buying a group gift. Explain why only part of the graph of the rational function is meaningful in the context of the problem.

STANDARDIZED TEST PRACTICE

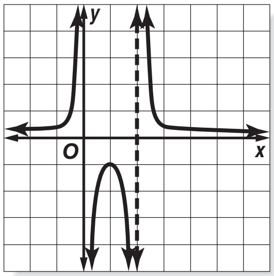
55. ACT/SAT Which set is the domain of the function graphed below?

A $\{x \mid x \neq 0, 2\}$

B $\{x \mid x \neq -2, 0\}$

C $\{x \mid x < 4\}$

D $\{x > -4\}$



56. REVIEW $\dfrac{x+2}{x+3} + \dfrac{4}{x^2+x-6} =$

F $\dfrac{-3x-9}{x^2+x-6}$

H $\dfrac{x^2}{x^2+x-6}$

G $\dfrac{x^2-3x-24}{x^2+x-6}$

J $\dfrac{x^2+x-1}{x^2+x-6}$

Spiral Review

Simplify each expression. (Lessons 8-1 and 8-2)

57. $\dfrac{3m+2}{m+n} + \dfrac{4}{2m+2n}$

58. $\dfrac{5}{x+3} - \dfrac{2}{x-2}$

59. $\dfrac{2w-4}{w+3} \div \dfrac{2w+6}{5}$

Find all of the rational zeros for each function. (Lesson 6-8)

60. $f(x) = x^3 + 5x^2 + 2x - 8$

61. $g(x) = 2x^3 - 9x^2 + 7x + 6$

62. ART Joyce Jackson purchases works of art for an art gallery. Two years ago she bought a painting for \$20,000, and last year she bought one for \$35,000. If paintings appreciate 14% per year, how much are the two paintings worth now? (Lesson 6-5)

Solve each equation by completing the square. (Lesson 5-5)

63. $x^2 + 8x + 20 = 0$

64. $x^2 + 2x - 120 = 0$

65. $x^2 + 7x - 17 = 0$

66. Write the slope-intercept form of the equation for the line that passes through $(1, -2)$ and is perpendicular to the line with equation $y = -\frac{1}{5}x + 2$. (Lesson 2-4)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each proportion.**

67. $\dfrac{16}{v} = \dfrac{32}{9}$

68. $\dfrac{7}{25} = \dfrac{a}{5}$

69. $\dfrac{6}{15} = \dfrac{8}{s}$

70. $\dfrac{b}{9} = \dfrac{40}{30}$